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Conjectured derivation of the Planck radiation spectrum from Casimir energies

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Abstract

Numerical calculation suggests that there is an intimate connection between the Planck radiation spectrum and Casimir energies. The Planck spectrum including zero-point radiation seems to satisfy a natural maximum-uniformity principle for Casimir energies whereas alternative choices of spectra do not. Specifically, we consider a set of identical conducting-walled boxes at the same temperature, but each has a conducting partition placed at a different location in the box, so that across the collection of boxes the partitions are uniformly spaced across the volume; then the Planck spectrum corresponds to that spectrum of random radiation satisfying the Wien displacement theorem (having constant energy $k_B T$ per normal mode at low frequencies and zero-point energy $(1/2)\hbar\omega$ per normal mode at high frequencies) which gives a monotonic change in Casimir energies with partition position and maximum uniformity for the Casimir energies across the collection of boxes. For simplicity, the analysis is presented for waves in one space dimension.

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1. Introduction

Suppose that we had a collection of conducting-walled boxes, each with a conducting partition which divided the box into two sections, where the boxes were identical except for the placement of the partition. Suppose further that each section in the boxes contained random radiation with the same fundamental spectrum which included both zero-point radiation and also some additional random radiation with a non-zero low-frequency limit. Now although the fundamental spectrum is the same in each box, each box will have a slightly different total radiation energy because the differing placements of the partitions will lead to different normal mode frequencies and therefore to different (Casimir) energies. It seems interesting to ask what spectrum would give the greatest uniformity of radiation energy across the collection of boxes despite the variation due to the partitions. Numerical calculation suggests that greatest uniformity is provided by the Planck spectrum. This seems a curious result which suggests an intimate connection between the Planck spectrum of thermal radiation and Casimir energies.

Thermal radiation is a fundamental thermodynamic system which holds a special place in the history of classical physics [1]. Although thermal radiation [2] can be described by a sum over harmonic oscillator modes, and therefore the thermodynamic restrictions on the adiabatic curves can be derived in the form of Wien's displacement theorem[1], the classical mode description does not give the full thermodynamic behaviour because it does not determine the thermodynamic entropy for each mode. Some additional idea of order, of uniformity, is required for the determination of the entropy function for each mode and hence for the determination of the fundamental spectrum of thermal radiation.

In connection with random radiation, twentieth century physics contributed two important ideas, zero-point energy and Casimir forces, which raise new possibilities for recognizing a natural idea of uniformity for thermal radiation. Zero-point energy is random energy which is present even at zero temperature[2]. Thermodynamics allows the possibility of zero-point energy and experimental evidence, such as van der Waals forces, requires its existence [4]. Casimir forces and energies are those which arise due to the discrete, classical normal modes structure of a system [5]. In total contrast to particles, waves are influenced in the interior of a volume by the presence of boundary conditions at the walls. Thus if a thin conducting partition is introduced into a conducting-walled box, then the energy of the system is changed due to the new boundary conditions at the conducting partition. The change in energy with the partition location is termed a Casimir energy. Casimir energies serve to couple total electromagnetic radiation energy in a partitioned box to the specific spectrum of random radiation. Thus if we consider a collection of identical conducting-walled containers, each with a partition at a different location, and each box having random radiation at the same temperature, then each of these boxes will have a different (average) thermal energy. And different assumed spectral distributions for thermal radiation will lead to different distributions of energies among the partitioned boxes. It is tempting to speculate that nature will choose as the spectrum of thermal radiation precisely that spectrum which gives greatest uniformity for the Casimir energies of the partitioned boxes. In the presence of zero-point radiation to prevent an 'ultraviolet catastrophe', numerical calculation suggests that the Planck spectrum satisfies this maximum-uniformity idea whereas other radiation spectra do not.

2. Thermodynamics of waves in a one-dimensional box

Although the calculations to be described below can be carried through for electromagnetic waves in a three-dimensional box, we will consider a thermodynamic wave system in one spatial dimension rather than in three, because the mathematics is distinctly simpler while the physical ideas are unchanged[3]. Thus we can imagine one-dimensional thermodynamic wave systems consisting of waves on a string, or of electromagnetic waves which are required to move between two conducting walls with wave vectors $\mathbf{k}$ which are always perpendicular to the walls.

[1] See, for example, M Planck in [1], pp 72–83 or [3].

[2] Physicists today usually regard zero-point radiation as a 'quantum' phenomenon. However, zero-point radiation can also be regarded as classical random radiation, just as thermal radiation was regarded as classical random radiation before 1900.

[3] In this paper we have discussed the case of waves in one spatial dimension. However, the same thermodynamic analysis applies immediately in three dimensions. The behaviour of Casimir forces for the Planck spectrum within a three-dimensional rectangular conducting box with a conducting partition is found numerically to repeat the same sort of axis-hugging behaviour as found in the one-dimensional case. See the curves in [6].

Systems satisfying the wave equation in a container with conducting walls can be described in terms of normal modes of oscillation, each of which corresponds to a harmonic oscillator system [7] with Lagrangian

$$L(q_\lambda, \dot{q}_\lambda) = \Sigma_\lambda (1/2)\big(\dot{q}_\lambda^2 - \omega_\lambda^2 q_\lambda^2\big) \tag{1}$$

where the q_λ are the amplitudes of the normal modes. For waves in one spatial dimension inside a box of length L, the normal modes can be labelled by a single integer index n where the associated frequency ω_n is given by $\omega_n = n\pi c/L, n = 1, 2, 3, \ldots,$ where c is the speed of the waves. Wien's displacement law, which follows from the application of the laws of thermodynamics to a harmonic oscillator system, tells us that the energy $\mathcal{U}$ of a normal mode at frequency ω and temperature T is given by

$$\mathcal{U}(\omega, T) = -\omega\phi'(\omega/T) \tag{2}$$

where ϕ' is a function of the single variable ω/T [8].

Wien's displacement law allows two limits which make the mode energy $\mathcal{U}$ independent of one of its two variables. Thus if $\phi' \to$ const when $\omega/T \gg 1$, then $\mathcal{U}$ depends upon ω alone. This corresponds to temperature-independent zero-point radiation

$$\mathcal{U} \to \mathcal{U}_{zp}(\omega) = (1/2)\hbar\omega \qquad \text{for} \quad \omega/T \gg 1 \tag{3}$$

where the constant $\hbar$ must be chosen to have the value of Planck's constant in order to fit with van der Waals forces [4]. On the other hand, if $\phi' \to \text{const}/(\omega/T)$, when $\omega/T \ll 1$, then $\mathcal{U}$ depends upon T alone. This corresponds to the Rayleigh–Jeans spectrum

$$\mathcal{U} \to \mathcal{U}_{RJ}(T) = k_B T \qquad \text{for} \quad \omega/T \ll 1 \tag{4}$$

which holds at low frequencies with k_B as Boltzmann's constant.

Since we are not interested in numerical calculations of thermodynamic quantities, we will use natural units[4] where $\hbar = 1$ so that frequency is measured in energy units, and $k_B = 1$ so that temperature is measured in energy units, and entropy is a pure number,

$$\mathcal{U}_{zp}(\omega) = (1/2)\omega \qquad \text{and} \qquad \mathcal{U}_{RJ}(T) = T. \tag{5}$$

It is convenient to introduce the thermal energy $\mathcal{U}_T(\omega, T)$ of a mode of frequency ω as the mode energy above the zero-point energy

$$\mathcal{U}_T(\omega, T) = \mathcal{U}(\omega, T) - \mathcal{U}_{zp}(\omega). \tag{6}$$

Although the total energy $\mathcal{U}$ is related to forces, only the thermal energy $\mathcal{U}_T$ influences the entropy[5].

The total thermal radiation energy U_T in a box of length L is given by the sum over the thermal energies $\mathcal{U}_T$ of the modes of frequencies $\omega_n = n\pi c/L$ for integer n. We consider only the thermal energy $\mathcal{U}_T$ since thermodynamics requires that U_T is a finite quantity when summed over all normal modes. In contrast, use of the modes' total energies $\mathcal{U}$ or zero-point energies $\mathcal{U}_{zp}$ will give a divergence in the sum over (infinitely many) high-frequency modes. In a one-dimensional box which is so large that the discrete sum over normal modes can be replaced by an integral, we can use (2) to obtain the total thermal energy $U_T(L, T)$ in the form

$$\begin{aligned} U_T(L, T) &= \sum_{n=1}^{\infty} \mathcal{U}_T\left(\frac{n\pi c}{L}, T\right) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[\phi'\left(\frac{n\pi c}{LT}\right) - \frac{1}{2}\right] \\ &\approx \int_0^\infty \mathrm{d}n\, \frac{n\pi c}{L}\left[\phi'\left(\frac{n\pi c}{LT}\right) - \frac{1}{2}\right] = \frac{L}{c\pi} T^2 \int_0^\infty \mathrm{d}z\, z \left[\phi'(z) - \frac{1}{2}\right] \end{aligned} \tag{7}$$

[4] See the discussion of natural units by Garrod ([8], p 120). The choice $\hbar = 1$ is familiar to particle physicists. The measurement of temperature in energy units is familiar in thermodynamics where our choice corresponds to the use of what is usually termed τ instead of T.

[5] See, for example, Boyer in [3].

for one space dimension. This is just the Stefan–Boltzmann result appropriate for one space dimension[6]. In the case of waves in three spatial dimensions, the frequencies of the normal modes ω_{lmn} would be labelled by three integer indices and the same procedure would lead to a T^4 temperature dependence for a large container.

3. Normal mode structure and Casimir forces

The Stefan–Boltzmann law in (7) gives the temperature dependence of the total thermal energy but provides no information regarding the spectrum of thermal radiation. Now in obtaining equation (7), we took the limit of a large box L and so replaced the sum over normal modes by an integral. However, by going to the continuum limit, we lost the information which might be available in the discrete spectrum of the normal modes. It was Casimir who saw the possibility of new forces and energies linked to this discreteness of the classical normal mode structure. The most famous example of such forces is the original Casimir calculation [5] of the force between conducting parallel plates arising from electromagnetic zero-point radiation. Casimir worked specifically with zero-point fields; however, the idea is not limited to zero-point radiation. Any spectrum of random classical radiation will lead to Casimir energies associated with the discrete classical normal mode structure of a container[7]. Indeed, every thermodynamic variable (energy, entropy, free energy, force) will depend upon the normal mode structure.

It should be emphasized how totally different this classical wave situation is from the classical particle situation of ideal gas particles in a box. Thus if a box with reflecting walls is filled with ideal gas particles at temperature T, then the introduction of a thin reflecting partition does not change the system energy and does not involve any average force on the partition. In total contrast, the introduction of a conducting partition into a conducting-walled box of thermal radiation leads to a change in the normal mode structure and hence both to position-dependent energy changes (Casimir energies) and to average forces on the partition (Casimir forces). These Casimir energies and forces will depend upon the precise spectrum of random radiation and upon the precise location of the partition. In this paper we note that the Planck spectrum for thermal radiation equilibrium can be obtained from a natural maximum-uniformity principle for the Casimir energy changes associated with the placement of partitions in boxes of radiation.

4. Change in Casimir energy due to a partition

We now consider a one-dimensional box of length L and calculate the change of radiation energy $\Delta U(x, L, T)$ with position x for a partition which is located a distance x from the left-hand end of the box, $0 \leqslant x \leqslant L$. The energy of each normal mode of frequency ω_n is given

[6] It is amusing to carry out Boltzmann's derivation for the one-dimensional case. We assume that the thermal energy and entropy of our waves in a very large one-dimensional box of length L satisfy $U_T(T, L) = Lu_T(T)$, and $S(T, L) = Ls(T)$ where the densities are functions of temperature alone. For a normally incident plane wave, we expect a pressure $p = \mathcal{E}/V$ rather than $p = (1/3)\mathcal{E}/V$. Multiplying by the area of the walls, the force on the bounding partition corresponds to $X = u$ where u is the energy per unit length. These are electromagnetic results which involve no thermodynamics. Now substituting into $T\,\mathrm{d}S(T, L) = \mathrm{d}U_T(T, L) + X_T\,\mathrm{d}L$ and separating differentials on both sides, we have $s = 2u_T/T$ and $\mathrm{d}s/\mathrm{d}T = (1/T)(\mathrm{d}u_T/\mathrm{d}T)$. Differentiating the equation for s with respect to temperature and substituting into the second, we find a differential equation with solution $u_T = \alpha T^2$ and so $s = 2\alpha T$ where α is an unknown constant.

[7] Casimir, in [5], gives the force per unit area between two conducting parallel plates due to electromagnetic zero-point radiation. The Rayleigh–Jeans spectrum gives a different force per unit area, $F/A = -\zeta(3)k_BT/(4\pi x^3)$. See, for example, [9].

by $\mathcal{U}(\omega_n, T)$. The partition changes the normal mode frequencies and so produces a position-dependent energy change $\Delta U(x, L, T)$ which is a Casimir energy. We will calculate the Casimir energy $\Delta U(x, L, T)$ as the change in the system energy when the partition is placed a distance x from the left-hand wall compared to when the partition is placed at $x = L/2$ in the middle of the box,

$$\begin{aligned}\Delta U(x, L, T) &= \{U(x, T) + U(L - x, T)\} - \{U(L/2, T) + U(L/2, T)\} \\ &= \left\{\sum_{n=1}^{\infty}\mathcal{U}\left(\frac{cn\pi}{x}, T\right) + \sum_{n=1}^{\infty}\mathcal{U}\left(\frac{cn\pi}{L-x}, T\right)\right\} - 2\sum_{n=1}^{\infty}\mathcal{U}\left(\frac{cn\pi}{L/2}, T\right). \qquad (8)\end{aligned}$$

5. Casimir energy for the zero-point spectrum

Equation (8) for the Casimir energy $\Delta U(x, L, T)$ of a box has been expressed as a sum over the total energy of each normal mode. Before we can discuss a maximum-uniformity principle involving this energy, we must know that it is well-defined. We have already noted that the sum over the thermal energy $\mathcal{U}_T(\omega, T)$ of the modes represents the total thermal energy U_T and is finite, while the sum including the zero-point energy $\mathcal{U}_{zp}(\omega)$ is divergent. However, the Casimir energy $\Delta U(x, L, T)$ in (8) can be defined as a limit and is finite. We recall that, in contrast to an ideal system, any physical wave system (such as a string with clamped ends or else electromagnetic fields in a region bounded by good conductors) will not enforce the normal mode structure at very high frequencies (short wavelengths). Thus it is natural to introduce a smooth cut-off $\exp(-\Lambda\omega/c)$ related to frequency $\omega = ck$

$$U(L, T, \Lambda) = \sum_{n=1}^{\infty}\mathcal{U}(\omega_n, T)\exp(-\Lambda\omega_n/c). \qquad (9)$$

Next we carry out the subtractions corresponding to (8) to obtain the Casimir energy, $\Delta U(x, L, T, \Lambda)$, and then allow the no-cut-off limit $\Lambda \to 0$. Although here we will work with an exponential cut-off because it is easy to sum the geometric series, the result is very general; any smooth cut-off function dependent on frequency alone will give the same result [10].

In this fashion, we can calculate the Casimir energy for the zero-point radiation spectrum in (5),

$$\begin{aligned}\Delta U_{zp}(x, L) &= \lim_{\Lambda\to 0}\left\{\sum_{n=1}^{\infty}\frac{1}{2}\frac{cn\pi}{x}\exp\left(-\Lambda\frac{n\pi}{x}\right) + \sum_{n=1}^{\infty}\frac{1}{2}\frac{cn\pi}{L-x}\exp\left(-\Lambda\frac{n\pi}{L-x}\right)\right. \\ &\quad \left. - 2\sum_{n=1}^{\infty}\frac{1}{2}\frac{cn\pi}{L/2}\exp\left(-\Lambda\frac{n\pi}{L/2}\right)\right\} \\ &= \lim_{\Lambda\to 0}\left\{-\frac{c}{2}\frac{\partial}{\partial\Lambda}\left[\frac{1}{\exp[\Lambda\pi/x]-1} + \frac{1}{\exp[\Lambda\pi/(L-x)]-1}\right.\right. \\ &\quad \left.\left. - 2\frac{1}{\exp[\Lambda\pi/(L/2)]-1}\right]\right\} \\ &= \lim_{\Lambda\to 0}\left\{\left[\frac{cx}{2\Lambda^2\pi} - \frac{c\pi}{24x} + \bigcirc(\Lambda)\right] + \left[\frac{c(L-x)}{2\Lambda^2\pi} - \frac{c\pi}{24(L-x)} + \bigcirc(\Lambda)\right]\right. \\ &\quad \left. - 2\left[\frac{c(L/2)}{2\Lambda^2\pi} - \frac{c\pi}{24(L/2)} + \bigcirc(\Lambda)\right]\right\} = -\frac{c\pi}{24}\left(\frac{1}{x} + \frac{1}{L-x} - \frac{2}{L/2}\right). \qquad (10)\end{aligned}$$

Thus we obtain the change in zero-point energy associated with the position x of the partition,

$$\Delta U_{zp}(x, L) = -\frac{c\pi}{24}\left(\frac{1}{x} + \frac{1}{L-x} - \frac{2}{L/2}\right). \tag{11}$$

The total Casimir energy at finite temperature T then involves

$$\Delta U(x, L, T) = \Delta U_T(x, L, T) + \Delta U_{zp}(x, L) \tag{12}$$

where ΔU, ΔU_T, ΔU_{zp} are formed from the respective mode energies $\mathcal{U}, \mathcal{U}_T$ and $\mathcal{U}_{zp}$. From the result (11) for zero-point energy, we see that $\Delta U(x, L, T)$ is finite for any spectrum $\mathcal{U}_T$ of thermal radiation which has finite total energy U_T.

6. Casimir energy for the Rayleigh–Jeans spectrum

The Wien displacement law gives zero-point radiation (3) as the possible high-frequency limit of thermal radiation. The low-frequency limit of the Wien law corresponds to the Rayleigh–Jeans spectrum (4). The Casimir energies associated with the Rayleigh–Jeans spectrum can again be calculated analytically making use of a high-frequency cut-off just as in the case of the zero-point spectrum. We have

$$\begin{aligned}\Delta U_{RJ}(x, L, T) = \lim_{\Lambda\to 0}&\left\{\sum_{n=1}^{\infty} T\exp\left(-\Lambda\frac{n\pi}{x}\right) + \sum_{n=1}^{\infty} T\exp\left(-\Lambda\frac{n\pi}{L-x}\right)\right.\\ &\left.- 2\sum_{n=1}^{\infty} T\exp\left(-\Lambda\frac{n\pi}{L/2}\right)\right\}\\ = \lim_{\Lambda\to 0}&\left\{\frac{T}{\exp[\Lambda\pi/x]-1} + \frac{T}{\exp[\Lambda\pi/(L-x)]-1} - 2\frac{T}{\exp[\Lambda\pi/(L/2)]-1}\right\}\\ = \lim_{\Lambda\to 0}&\left\{T\left[\frac{x}{\Lambda\pi} - \frac{1}{2} + \frac{\pi\Lambda}{12x} - \cdots\right] + T\left[\frac{L-x}{\Lambda\pi} - \frac{1}{2} + \frac{\pi\Lambda}{12(L-x)} - \cdots\right]\right.\\ &\left.- 2T\left[\frac{L/2}{\Lambda\pi} - \frac{1}{2} + \frac{\pi\Lambda}{12(L/2)} - \cdots\right]\right\} = 0.\end{aligned} \tag{13}$$

Thus we find that the Rayleigh–Jeans spectrum gives no Casimir energy changes at all. Indeed, the Rayleigh–Jeans spectrum is the unique spectrum which produces no Casimir energy changes associated with the placement of the Casimir partition, $\Delta U_{RJ}(x, L, T) = 0$.[8]

7. Casimir energies for various radiation spectra

Corresponding to any spectrum of random radiation, we can calculate the corresponding Casimir energies. In one spatial dimension, it is quick to evaluate the Casimir energies for various spectral functions $\mathcal{U}(\omega, T)$ on a home computer. One separates out the divergent zero-point energy contribution corresponding to (11) and then evaluates the thermal contribution to the Casimir energy $\Delta U_T(x, L, T)$ as in (8) for any assumed thermal spectrum $\mathcal{U}_T(\omega, T) = \mathcal{U}(\omega, T) - \mathcal{U}_{zp}(\omega)$. The total Casimir energy $\Delta U(x, L, T)$ is the sum (12) of the thermal contribution and the zero-point contribution (11).

[8] There are, however, Casimir forces and changes in Helmholtz free energy. One also finds interesting temperature-independent entropy changes with partition position $\Delta S_{RJ}(x, L, T) = (1/2)\,|\ln[x/(L-x)]|$. This seems reminiscent of temperature-independent changes associated with the mixing entropy of ideal gas particles. Similar changes have been noted in quantum field theory in [11].

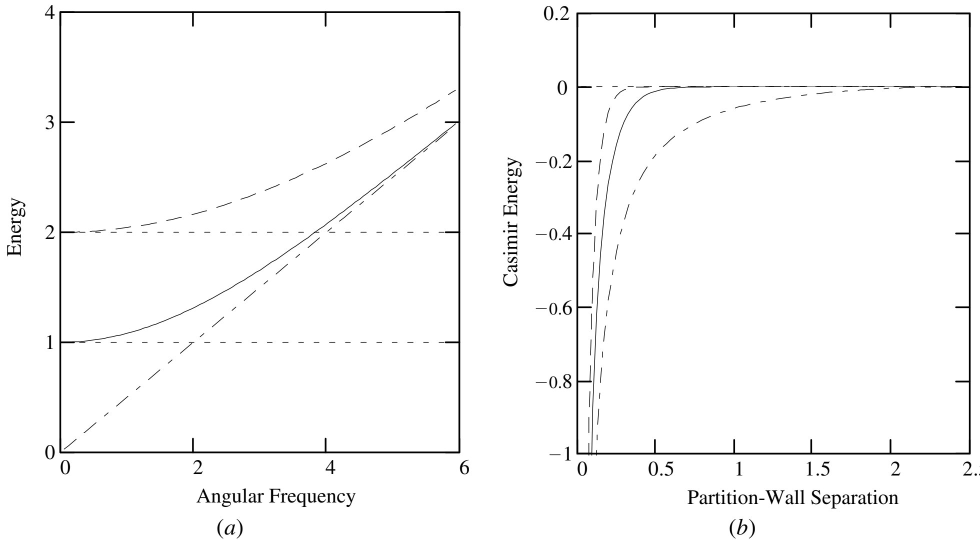

Figure 1. (*a*) *Planck spectrum energy*. The energy $\mathcal{U}$ per normal mode is plotted versus mode angular frequency ω. The solid curve corresponds to the Planck spectrum with zero-point radiation $\mathcal{U}_P(\omega, T) = (\omega/2)\coth(\omega/2T)$ at temperature $T = 1$, and the dashed curve is at $T = 2$. The dash-dot sloping curve corresponds to the zero-point spectrum $\mathcal{U}_{zp}(\omega) = \omega/2$. The horizontal dotted curves at energies $\mathcal{U} = 1$ and $\mathcal{U} = 2$ correspond to the Rayleigh–Jeans spectrum for temperatures $T = 1$ and $T = 2$. (*b*) *Planck Casimir energy*. The Casimir energies ΔU are plotted versus partition-wall separation x for the spectra $\mathcal{U}$ in (*a*) in a box of length $L = 5$. The functions are symmetric about the box midpoint $x = L/2 = 2.5$. The solid curve is the Casimir energy for the Planck spectrum with zero-point radiation at $T = 1$ and the dashed curve at $T = 2$. The dash-dot curve is the zero-point Casimir energy, and the dotted curve at $\Delta U = 0$ is the Rayleigh–Jeans Casimir energy at any temperature. At small partition-wall separations, the Planck Casimir energy follows the zero-point curve, but then goes over to the Rayleigh–Jeans curve at larger separations. At higher temperatures, the Planck curve moves to the Rayleigh–Jeans limit at smaller values of x.

In figure 1(*a*), we show the spectral energies $\mathcal{U}(\omega, T)$ for zero-point radiation $\mathcal{U}_{zp}(\omega) = (1/2)\omega$, for the Rayleigh–Jeans spectrum $\mathcal{U}_{RJ}(\omega, T) = T$, and for the Planck spectrum $\mathcal{U}_P(\omega, T) = (1/2)\omega\coth(\omega/2T)$ at temperatures $T = 1$ and $T = 2$. The associated Casimir energies $\Delta U_{zp}(x, L)$, $\Delta U_{RJ}(x, L, T)$, $\Delta U_P(x, L, T)$ are shown in figure 1(*b*) for a one-dimensional box of length $L = 5$ for the same temperatures $T = 1$, and $T = 2$. The Casimir energy $\Delta U_P(x, L, T)$ for the Planck spectrum rises from the zero-point Casimir energy $\Delta U_{zp}(x, L)$ at small partition-wall separations x and then hugs the x-axis corresponding to the Rayleigh–Jeans Casimir energy $\Delta U_{RJ}(x, L, T) = 0$ at larger separations x from the nearest wall. The Casimir energy is symmetric around the centre of the box at $x = L/2 = 2.5$. The Planck-spectrum curves move monotonically in both (nearest) partition-wall separation x for fixed temperature T and in T for fixed x, moving from the zero-point spectrum Casimir energy at small separations and low temperatures over to agreement with the Rayleigh–Jeans spectrum Casimir energy at larger separations and high temperatures. One cannot help but wonder to what extent such behaviour is special to the Planck spectrum or is a common feature of any spectra satisfying the Wien displacement law (2).

In order to answer this question, we consider various radiation spectra satisfying the Wien displacement law with the Rayleigh–Jeans spectrum and zero-point radiation as the respective low- and high-frequency limits. The roughest possible interpolation spectrum between the Rayleigh–Jeans spectrum at low frequency and the zero-point radiation at high frequency is

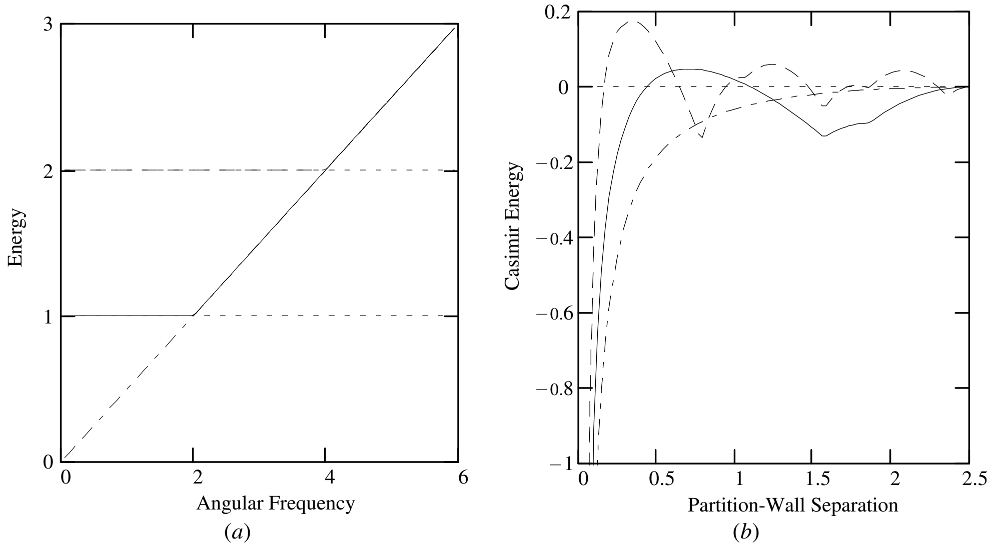

Figure 2. (*a*) *Rough thermal spectrum*. The energy $\mathcal{U}$ per normal mode is plotted versus mode angular frequency ω. The solid curve corresponds to the rough approximation $\mathcal{U}_1(\omega, T)$ in equation (14) for a thermal radiation spectrum at $T = 1$, and the dashed curve is the rough spectrum at $T = 2$. The rough spectrum satisfies the Wien displacement law and limits, following the Rayleigh–Jeans spectrum at low frequency $0 \leqslant \omega \leqslant 2T$, and then the zero-point spectrum at high frequency $2T \leqslant \omega$. The dash-dot sloping curve corresponds to the zero-point spectrum. The horizontal dotted curves at energies $\mathcal{U} = 1$ and $\mathcal{U} = 2$ correspond to the Rayleigh–Jeans spectrum for temperatures $T = 1$ and $T = 2$. (*b*) *Rough spectrum Casimir energy*. The Casimir energies ΔU are plotted versus partition-wall separation x for the spectra $\mathcal{U}$ in (*a*) in a box of length $L = 5$. The solid curve gives the Casimir energy for the rough thermal radiation spectrum $\mathcal{U}_1(\omega, T)$ in equation (14) at $T = 1$, and the dashed curve is for the rough spectrum Casimir energy at $T = 2$. The dash-dot curve is the zero-point Casimir energy, and the dotted curve at $\Delta U = 0$ is the Rayleigh–Jeans Casimir energy at any temperature. At small partition-wall separations x, the rough-spectrum Casimir energy follows the zero-point Casimir curve, but goes towards the Rayleigh–Jeans curve at larger separations. The Casimir energies for the rough thermal spectrum are not monotonic in x for fixed T or in T for fixed x. Also, at larger separations, the curves do not hug the horizontal axis to the extent of the Planck spectrum shown in figure 1(*b*).

given by

$$\mathcal{U}_1(\omega, T) = T\left(1 - \frac{\omega}{2T}\right)\theta\left(1 - \frac{\omega}{2T}\right) + \frac{1}{2}\omega \tag{14}$$

where θ is the Heaviside step function. This corresponds to starting at low frequency ω with a flat energy spectrum at T until the frequency reaches $\omega = 2T$ where the zero-point spectrum $(1/2)\omega$ takes over. This spectrum is shown in figure 2(*a*) for temperatures $T = 1$ and $T = 2$, and the corresponding Casimir energies are plotted as a function of x in figure 2(*b*) for a box of length $L = 5$. We see that the Casimir energies rise from the diverging zero-point Casimir energy at small values of x and then go towards zero at large values of x. However, the Casimir energies for this rough approximation do not provide monotonic transition curves in x for fixed T or in T for fixed x between the zero-point and Rayleigh–Jeans Casimir energies; rather the Casimir energies cross and recross the low- and high-temperature limits and change sign.

An interpolation of greater smoothness is provided by a quadratic spectral form

$$\mathcal{U}_2(\omega, T) = T\left(1 - \frac{\omega}{2T} + \frac{1}{16}\frac{\omega^2}{T^2}\right)\theta\left(1 - \frac{\omega}{4T}\right) + \frac{1}{2}\omega. \tag{15}$$

The corresponding Casimir energies are smoother and stay closer to the high-temperature result, but again the curves are not monotonic in x and T.

The Casimir energies for a large number of other spectral forms were evaluated numerically, but none gave the uniformity of the Planck spectrum; either the Casimir energy was not monotonic in x and T, or else the Casimir energy did not hug the axis as well as did the Planck spectrum. Numerical calculation suggests that indeed the Planck spectrum has a special relationship with Casimir energies.

8. Maximum uniformity principle for thermal radiation

We suggest that the Planck spectrum satisfies a maximum-uniformity principle for Casimir energies. We consider a collection of conducting-walled boxes of length L which differ only in the placement of the partition. Our ideas of uniformity suggest that the collection should have the partitions distributed uniformly in x along the open interval $(0, L)$. If each of these boxes contained thermal radiation at the same temperature T, then the boxes would still differ in their total energies because of the presence of the partitions at different locations and the associated difference in Casimir energies. We suggest that nature would choose the spectrum satisfying the Wien displacement theorem which gives maximum uniformity of energy among the boxes.

We consider all spectral functions $f(\omega/T)$ corresponding to the Wien law

$$\mathcal{U}(\omega, T) = T f(\omega/T) \tag{16}$$

where

$$f(z) = 1 + O(z^2) \qquad \text{for small } z \quad f(z) \geqslant \frac{1}{2}z \qquad \int_0^\infty \mathrm{d}z \left[f(z) - \frac{1}{2}z \right] < \infty. \tag{17}$$

These restrictions give the Rayleigh–Jeans limit at low frequency, positive thermal energy $\mathcal{U}(\omega, T) = \mathcal{U}_T(\omega, T) + \mathcal{U}_{zp}(\omega) \geqslant \mathcal{U}_{zp}(\omega)$ at all frequencies and finite thermal energy U_T when summed over all modes. We require that the Casimir energies arising from the radiation spectrum $f(\omega/T)$ must move monotonically with both partition-wall separation x for fixed temperature T, and for T at fixed x. Finally, we calculate the Casimir energy $\Delta U(x, L, T)$ in (8) using (11) and (12) with any function f in (16) satisfying the criteria of (17). Then the departure I_N from a uniform energy among the partitioned boxes is given by

$$I_N = \sum_{i=1}^{N} |\Delta U(x_i, L, T)| \tag{18}$$

where we have N boxes with partitions spaced uniformly in the open interval $(0, L)$ at $x_i = iL/(N+1)$, $i = 1, 2, \ldots, N$. Thus the first box has its partition at x_1, the second box has its partition at x_2, etc. The placement of the partitions in the open interval is necessitated by the meaning of a partitioned box and by the divergence of the zero-point energy $\Delta U_{zp}(x, L, T)$ in (11) at the ends of the box when $x = 0$ or $x = L$. In order to obtain a reliable sampling over the possible positions of the partitions, we will want to take the number N of boxes very large so that the spacing interval between boxes $\delta_N = x_{i+1} - x_i = L/(N+1)$ is much less than both the length L of each box and the length c/T associated with thermal radiation at temperature T. Indeed, we consider a sequence of sums $I_1, I_2, \ldots, I_N$ with $N \to \infty$, and find the radiation spectrum f_N which makes I_N a minimum. For large N, the functions f_N converge to a unique function f which is the desired function giving maximum uniformity. Because a sum over a function at equally-spaced intervals Δx is directly related to an integral, this procedure is equivalent to integrating $|\Delta U(x, L, T)|$ over the open interval from $x_1 = \delta_N = L/(N+1)$ to $x_N = L - \delta_N$, and obtaining the function f_{δ_N} giving a minimum for the integral

$$I_{\delta_N} = \int_{x=\delta_N}^{x=L-\delta_N} \mathrm{d}x |\Delta U(x, L, T)|. \tag{19}$$

We then consider the limiting function

$$f = \lim_{\delta_N \to 0} f_{\delta_N}. \tag{20}$$

The maximum-uniformity principle states that nature will choose the limiting function $f(x)$ in (16) satisfying the criteria in (17) which gives monotonic Casimir energies in x and T and which makes the integral I in (19) a minimum. In practice, all we need to do is choose a value of $\delta \ll L$ and $\delta \ll c/T$; then the minimizing function f_δ will be independent of δ as $\delta \to 0$.

9. Euler–Maclaurin formula and scaling at infinite box length

The use of a finite box length L makes numerical evaluations easy but complicates attempts at analytic analysis. If we go to the limit of infinite box length $L \to \infty$ while holding the partition-wall separation x fixed, then the expression for the Casimir energy can be written in a form which scales with the temperature T. Thus rather than viewing a series of curves for different temperatures as in figures 1 and 2, we can consider a single curve which joins the zero-point and Rayleigh–Jeans limits for the Casimir energy. In order to avoid mathematical questions of convergence, we again separate out the zero-point and thermal contributions $U(x, T) = U_T(x, T) + U_{zp}(x)$ for the energy in a box. The connection between sums and integrals is given by the Euler–Maclaurin formula [12] which takes the form

$$\begin{aligned} U_T(x, T) &= \sum_{n=1}^{\infty} \mathcal{U}_T\left(\frac{n\pi c}{x}, T\right) = -T + \sum_{n=0}^{\infty} \mathcal{U}_T\left(\frac{n\pi c}{x}, T\right) \\ &= -T + \frac{x}{\pi c}\int_0^\infty \mathrm{d}z\, \mathcal{U}_T(z, T) + \frac{1}{2}[0 + T] \\ &\quad + \sum_{s=1}^{m} \frac{B_{2s}}{(2s)!}\left(\frac{\pi c}{x}\right)^{2s-1}\left[0 - \left(\frac{\partial^{2s-1}\mathcal{U}_T(\omega, T)}{\partial \omega^{2s-1}}\right)_{\omega=0}\right] \\ &\quad + \left(\frac{\pi c}{x}\right)^{2m-1}\int_0^\infty \mathrm{d}z\, P_{2m+1}(z)\frac{\partial^{2m+1}\mathcal{U}_T(z, T)}{\partial z^{2m+1}} \end{aligned} \tag{21}$$

where

$$P_{2m+1}(z) = (-1)^{m-1}\sum_{k=1}^{\infty}\frac{2\sin(2kxz/c)}{(2k\pi)^{2m+1}} \tag{22}$$

and the constants B_{2s} are the Bernoulli numbers, $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, \ldots$. Analogous forms hold for $U(L - x, T)$ and $U(L/2, T)$. We note that the expression (21) involves inverse powers of x for the last terms. In the limit of very large box-length L, the expressions for $U(L - x, T)$ and $U(L/2, T)$ simplify to the leading terms. Thus the Casimir energy for an infinite box with a partition at a fixed distance x from the left-hand wall becomes

$$
\begin{aligned}
\Delta U(x, L \to \infty, T) &= \Delta U_T(x, L \to \infty, T) + \Delta U_{zp}(x, L \to \infty, T) \\
&= \left\{ \sum_{n=1}^{\infty} \mathcal{U}_T\left(\frac{n\pi c}{x}, T\right) + \left(-\frac{1}{2}T + \frac{L-x}{\pi c}\int_0^{\infty} \mathrm{d}z\, \mathcal{U}_T(z, T)\right) \right. \\
&\quad \left. - 2\left(-\frac{1}{2}T + \frac{L/2}{\pi c}\int_0^{\infty} \mathrm{d}z\, \mathcal{U}_T(z, T)\right) \right\} - \frac{c\pi}{24}\frac{1}{x} \\
&= \sum_{n=0}^{\infty} \mathcal{U}_T\left(\frac{n\pi c}{x}, T\right) - \frac{xT}{\pi c}\int_0^{\infty} \mathrm{d}z\, \mathcal{U}_T(z, T) - \frac{1}{2}T - \frac{c\pi}{24}\frac{1}{x}.
\end{aligned}
\tag{23}
$$

Now the Wien displacement law in (16) gives $\mathcal{U}(\omega, T) = -\omega\phi'(\omega/T) = Tf(\omega/T)$. Therefore the quotient $\mathcal{U}(\omega, T)/T = f(\omega/T)$ is a function of ω/T only, and the Casimir energy quotient $\Delta U(x, L \to \infty, T)/T$ in (23) is a function of xT only

$$
\Delta U(x, L \to \infty, T)/T = \sum_{n=0}^{\infty} f_T\left(\frac{n\pi c}{xT}\right) - \frac{xT}{\pi c}\int_0^{\infty} \mathrm{d}z\, f_T(z) - \frac{1}{2} - \frac{c\pi}{24}\frac{1}{xT}
\tag{24}
$$

where $f_T(z) = f(z) - (1/2)z$. Thus in the case of an infinite box $L \to \infty$, the scaled Casimir energy $\Delta U(x, L \to \infty, T)/T$ depends upon the scaled distance xT through the one unknown function $f(\omega/T)$ of the scaled frequency ω/T where $f(\omega/T) = 1 + O(\omega^2/T^2)$ as $\omega/T \to 0$ and $f(\omega/T) \to (1/2)\omega/T$ as $\omega/T \to \infty$.

10. Casimir energies for various spectra for infinite box length

At small xT values, the sum in equation (24) will involve widely spaced arguments in $\pi c/(xT)$. However, $f_T(z) \to 0$ for large argument z. Thus at small xT values, we expect the Casimir energy to follow the zero-point curve

$$
\Delta U(x, L \to \infty, T)/T \to -\frac{\pi c}{24}\frac{1}{xT} + \frac{1}{2} + O(xT) \qquad \text{for} \quad xT \ll 1.
\tag{25}
$$

For larger values of xT, we use the Euler–Maclaurin expansion. Jeffreys and Jeffreys suggest [13] that the Euler–Maclaurin formula provides the best approximation in the sense of an asymptotic expansion between the sum and integral appearing in the expression (24). Thus we have

$$
\begin{aligned}
\Delta U(x, L \to \infty, T)/T &= \sum_{n=0}^{\infty} f_T\left(\frac{n\pi c}{xT}\right) - \frac{xT}{\pi c}\int_0^{\infty} \mathrm{d}z\, f_T(z) - \frac{1}{2} - \frac{c\pi}{24}\frac{1}{xT} \\
&= \left\{ \frac{1}{2}f_T(0) + \sum_{s=1}^{m} \frac{B_{2s}}{(2s)!}\left(\frac{\pi c}{xT}\right)^{2s-1}\left[-f_T^{2s-1}(0)\right] \right. \\
&\quad \left. + \left(\frac{\pi c}{xT}\right)^{2m-1}\int_0^{\infty} \mathrm{d}z\, P_{2m+1}(z) f_T^{(2m+1)}(z) \right\} - \frac{1}{2} - \frac{c\pi}{24}\frac{1}{xT} \\
&= \sum_{s=2}^{m} \frac{B_{2s}}{(2s)!}\left(\frac{\pi c}{xT}\right)^{2s-1}\left[-f_T^{2s-1}(0)\right] + \left(\frac{\pi c}{xT}\right)^{2m-1}\int_0^{\infty} \mathrm{d}z\, P_{2m+1}(z) f_T^{(2m+1)}(z)
\end{aligned}
\tag{26}
$$

where we have used the fact that $f_T(0) = 1$, $f_T^{(1)}(0) = -1/2$ and $B_2 = 1/6$. The terms in the last line of (26) involve inverse powers of xT and so vanish as $O[(xT)^{-3}]$ or faster. Thus for large xT, we recover the Casimir energy for the Rayleigh–Jeans spectrum

$$
\Delta U(x, L \to \infty, T)/T \to 0 \qquad \text{for} \quad xT \gg 1.
\tag{27}
$$

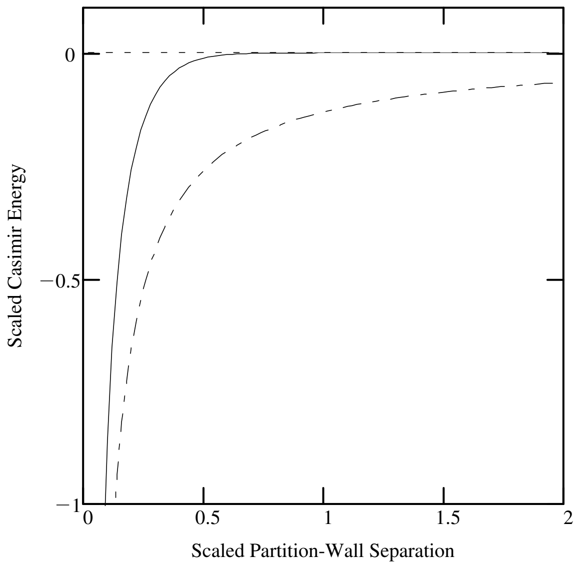

Figure 3. *Planck scaled Casimir energy*. The scaled Casimir energy $\Delta U/T$ is plotted versus the scaled partition-wall separation xT in a box of infinite length. The solid curve gives the scaled Casimir energy $\Delta U_P/T$ for the Planck spectrum with zero-point radiation. The dash-dot curve shows the scaled Casimir energy $\Delta U_{zp}/T$ for zero-point radiation, and the dotted curve $\Delta U_{RJ}/T = 0$ gives the Rayleigh–Jeans result.

The scaled Planck spectrum $\mathcal{U}_P(\omega, T)/T = f_P(\omega/T) = (\omega/(2T))\coth(\omega/(2T))$ can be expanded about $\omega/T = 0$ giving

$$f_P\left(\frac{\omega}{T}\right) = \frac{\omega}{2T}\coth\left(\frac{\omega}{2T}\right) = \sum_{n=0}^{\infty}\frac{B_n}{n!}\left(\frac{\omega}{2T}\right)^n + \frac{1}{2}\frac{\omega}{T}$$
$$= 1 + \frac{1}{12}\left(\frac{\omega}{T}\right)^2 - \frac{1}{720}\left(\frac{\omega}{T}\right)^4 + \frac{1}{30\,240}\left(\frac{\omega}{T}\right)^6 - \cdots. \tag{28}$$

Here the B_n are the same Bernoulli numbers as appear in the Euler–Maclaurin formula. We see that in (28) all the odd derivatives f_P^{2s+1} of f_P vanish at $\omega/T = 0$ so that the only correction term for the Euler–Maclaurin expansion (26) of the Planck Casimir energy is the remainder term

$$\Delta U_P(x, L \to \infty, T)/T = \left(\frac{\pi c}{xT}\right)^{2m-1}\int_0^{\infty} \mathrm{d}z\, P_{2m+1}(z) f_P^{(2m+1)}(z) \tag{29}$$

where P_{2m+1} is given in (22) and m is any integer greater than 1. We have used the fact that for derivatives above the first, $f_{PT}^{(n)}(z) = f_P^{(n)}(z)$. Form (28) for the Euler–Maclaurin expansion suggests that for large $xT \to \infty$, the scaled Casimir energy $\Delta U_P(x, L \to, T)/T$ from the Planck spectrum vanishes faster than any inverse power of (xT). The scaled Casimir energies for the zero-point spectrum, for the Rayleigh–Jeans spectrum and for the Planck spectrum are shown in figure 3.

Although it is easy to evaluate numerically the scaled Casimir energy for any spectrum $f(\omega/T)$ satisfying the limits (17) suggested by the Wien law, we will illustrate what we believe are the important aspects of other spectra by introducing a parameter b into a functional form related to the Planck spectrum,

$$\mathcal{U}_b(\omega, T, b) = T\frac{(b\omega/T)\exp[-(b+1)(\omega/2T)]}{1 - \exp[-b(\omega/T)]} + \omega/2. \tag{30}$$

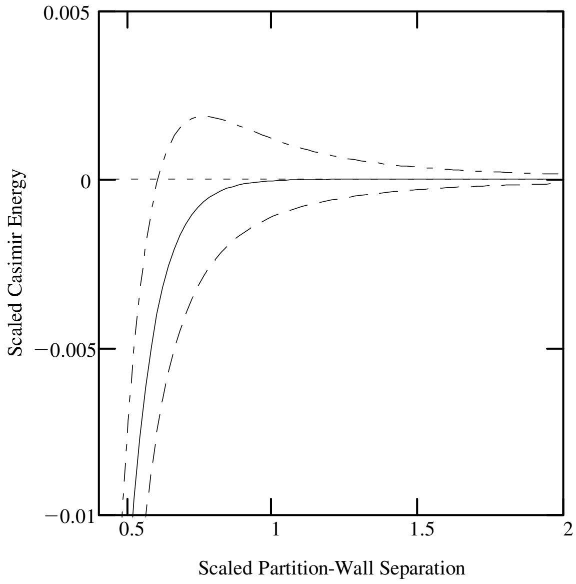

Figure 4. *Various scaled Casimir energies.* The scaled Casimir energy $\Delta U/T$ is plotted versus the scaled partition-wall separation xT for the parametrized spectrum $\mathcal{U}_b$ given in equation (30) for $b = 0.9$, $b = 1.0$ and $b = 1.1$. The solid curve for $b = 1$ corresponds to the Planck spectrum with zero-point radiation and is the same as in figure 3, except that the scale of the figure is quite different. The dash-dot curve gives the scaled Casimir energy for $b = 1.1$; this curve changes sign and is not monotonic in xT. The dashed curve corresponds to $b = 0.9$; this is monotonic in xT but does not approach the dotted Rayleigh–Jeans result $\Delta U_{RJ} = 0$ nearly as rapidly as does the solid Planck curve.

For any $b > 0$, this spectrum goes over to the Rayleigh–Jeans form at low frequency ω, and over to zero-point radiation at high frequency ω. For the choice $b = 1$, this becomes exactly the Planck law (28). If we expand the expression for $\mathcal{U}_b(\omega, T, b)$ in (30) as a power series about $\omega/T = 0$, we find

$$\begin{aligned}\mathcal{U}_b(\omega, T)/T = f_b(\omega/T) &= 1 + \left(\frac{-1}{24}b^2 + \frac{1}{8}\right)\left(\frac{\omega}{T}\right)^2 \\ &\quad + \frac{1}{48}(b-1)(b+1)\left(\frac{\omega}{T}\right)^3 + \left(\frac{7}{5760}b^4 - \frac{1}{192}b^2 + \frac{1}{384}\right)\left(\frac{\omega}{T}\right)^4 \\ &\quad + \frac{-1}{11\,520}(b-1)(b+1)(7b^2-3)\left(\frac{\omega}{T}\right)^5 + \cdots. \end{aligned} \qquad (31)$$

We note that each of the odd-order terms in ω/T has a factor of $(b-1)(b+1)$. Thus for a general choice of parameter b, this spectrum has odd powers of ω/T and hence nonvanishing odd-order derivatives at $\omega/T = 0$ which will appear in the Euler–Maclaurin expansion (26) for the Casimir energy. Thus for all values of the parameter b different from $b = 1$, we expect the Casimir energy arising from (30) to approach zero as some inverse power of xT.

The rescaled Casimir energy $\Delta U_b(x, L \to \infty, T)/T$ is plotted in figure 4 for $b = 0.9$, $b = 1.0$ and $b = 1.1$. (Note the scale change from figure 3.) The curve for $b = 1$ corresponds to the Planck spectrum with the expansion (28). We see that for $b = 1.1$, the reduced Casimir energy changes sign and hence is not a monotonic function of x for fixed T or of T for fixed x. This corresponds to the coefficient $(1/48)(b-1)(b+1) = 0.004\,375$ for the cubic term in the expansion (31) taking a positive value and so contributing a term $(B_4/4!)(\pi c/xT)^3 f^{(3)}(0)$ in the Euler–Maclaurin expansion. Thus the Casimir energy approaches the axis as $\text{const}(xT)^{-3}$. On the other hand for $b = 0.9$, the reduced Casimir energy does not change sign, but now it does not hug the axis so closely as does the Planck spectrum. This corresponds to the coefficient $(1/48)(b-1)(b+1) = -0.003\,958$ for the cubic term in the expansion (31) taking a negative value and so contributing a term $(B_4/4!)(\pi c/xT)^3 f^{(3)}(0)$ in the Euler–Maclaurin expansion. In this case the Casimir energy approaches the axis as $-\text{const}(xT)^{-3}$ with opposite sign from the approach for $b = 1.1$.

For those functions $f(\omega/T)$ which do give monotonic Casimir energies in xT, we evaluate the departure of the Casimir energy from uniformity

$$I_N = \int_{xT=1/N}^{N} \mathrm{d}(xT) \left| \frac{\Delta U(x, L \to \infty, T)}{T} \right| \tag{32}$$

and find the function $f_N(\omega/T)$ giving the smallest value for I_N. In the limit $N \to \infty$, we expect $f(\omega/T) = \lim_{N\to\infty} f_N(\omega/T)$ to correspond to the spectrum of 'greatest uniformity'. Our numerical calculations suggest that spectra behave generally as illustrated by the spectrum $\mathcal{U}_b(\omega, T)$; either the associated Casimir energy is not monotonic, or else the Casimir energy approaches the axis as an inverse power of xT, does not hug the axis tightly as the Planck spectrum, and so gives a larger value for I_N in (32). Analysis from the Euler–Maclaurin expansion together with numerical calculation suggests that the spectrum of greatest uniformity is provided by the Planck spectrum. We conjecture that definitive analytic calculation would confirm this special role for the Planck spectrum.

11. 'Ultraviolet catastrophe' without zero-point radiation

We should emphasize that our maximum-uniformity principle indeed requires the presence of the zero-point radiation energy. If no zero-point energy were present, then we would still require that the thermal spectrum $\mathcal{U}_T(\omega, T)$ gives energy equipartition at low frequency and goes to zero at high frequency so as to give a finite energy density for thermal radiation. For this case, the thermal energy would be the total energy used in (19) or (32). However, there would be no natural high-frequency limit. If we tried a smooth spectrum such as $\mathcal{U}_{CT}(\omega, T) = T\exp[-C(\omega/T)^2]$ with an adjustable parameter C but without zero-point energy, then we would find that the test integrals given in equations (19) and (32) decrease as the parameter C decreases, bringing the spectrum ever closer to the Rayleigh–Jeans spectrum, in which limit the integral vanishes $I = 0$ and there are no Casimir energy changes. The absence of any natural cut-off frequency represents behaviour reminiscent of the 'ultraviolet catastrophe' emphasized by Einstein and named by Ehrenfest in 1911. What prevents the catastrophic shift of thermal radiation to ever-higher frequencies is precisely the presence of zero-point radiation.

12. Simplicity for the Planck spectrum and for zero-point radiation

What basic, simple property distinguishes the Planck spectrum of blackbody radiation and makes it stand out amongst all other random radiation spectra? Recall that we know some of the basic, simple properties that distinguish the zero-point radiation spectrum; the zero-point spectrum is the unique spectrum of random radiation which is invariant under adiabatic compression, is scale invariant, and gives a Lorentz-invariant (indeed conformal-invariant) electromagnetic radiation spectrum [14]. But what is special about the Planck spectrum? In this paper we suggest one simple answer. The Planck spectrum is the random radiation spectrum which gives the greatest uniformity in Casimir energies for partitioned boxes when zero-point radiation is present.

We note that it is the presence of zero-point radiation which prevents the traditional classical ultraviolet catastrophe, and it is the non-zero parameter $\hbar$ setting the scale of zero-point radiation which is inherited by the Planck spectrum. The strong connection between thermal radiation and zero-point radiation has been noted many times in the past, especially in classical derivations of the Planck spectrum [15]. For example, a harmonic electric dipole oscillator of natural frequency ω_0 undergoing uniform acceleration a through zero-point radiation $\mathcal{U}_{zp}(\omega) = (1/2)\hbar\omega$ has an average energy $\mathcal{U}_{\text{oscillator}}(\omega_0, a)$ corresponding to the Planck spectrum with zero-point radiation $\mathcal{U}_{\text{oscillator}}(\omega_0, a) = \mathcal{U}_P(\omega_0, T)$ at temperature $T = \hbar a/2\pi c k_B$ [16]. In this case of uniform acceleration, it is emphatically clear that the Planck spectral form is inherited from zero-point radiation. Indeed, it seems likely that there is a group-theoretical connection through the conformal group.

13. Concluding summary

In this analysis, we have treated the thermodynamics of waves in one spatial dimension only. However, the ideas can be carried over to electromagnetic waves in three spatial dimensions [6]. Although the Wien displacement theorem reflects the information from adiabatic energy changes of the known harmonic oscillator Lagrangian for the wave modes of thermal radiation, the entropy associated with each mode is undetermined. Traditional classical statistical mechanics involving equally probable boxes on phase space does not find it possible to recognize a situation of natural maximum uniformity, of maximum entropy, for thermal radiation which avoids the divergent ‘ultraviolet catastrophe’. However, the use of Casimir energies, which connect different radiation spectra to different total radiation energies in a partitioned box, allows one to find a new maximum-uniformity criterion for radiation. In the absence of zero-point radiation, the new maximum-uniformity criterion recovers only the Rayleigh–Jeans spectrum. In the presence of zero-point radiation, numerical calculation indicates that the spectrum of maximum uniformity is the Planck spectrum. The Planck spectrum seems to be closely related to Casimir energies and zero-point radiation.

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